Beams on Elastic Foundation

Beams on elastic foundation, such as that in Figure 1, appear in building foundations, floating structures, beams that rest on a grid of perpendicular beams, and elsewhere. It also turns out that the governing differential equation is similar to that of cylindrical shells and tapered beams with curved webs. As a result, this material has quite broad applicability. However, it is important to exercise caution when applying this theory to problems where the foundation does not behave in a linear fashion, such as soils that respond nonlinearly to pressure or ships whose waterline area changes with sinkage.

In this document the beam is assumed to be an ordinary Euler-Bernoulli beam, thus the theory from that document carries over to this one. Accordingly, downwards distributed load on the beam is referred to as $q$, while it is denoted $q_z$ when it acts in the opposite direction, i.e., in the $z$-direction.

![Figure 1: Beam on elastic foundation.](image)

Foundation Stiffness

The foundation stiffness, $k_s$, which is illustrated in Figure 1, is conceptually straightforward. When the beam displaces downwards, the foundation exerts an upwards force, illustrated by the springs in Figure 1. $k_s$ has units of force per unit length along the beam. Another stiffness, denoted $k_\phi$, is also present, relating to rotation $\phi$ of the beam cross-section around the longitudinal axis of the beam. In other words, $k_\phi$ provides resistance against torsion of the beam. Techniques to determine $k_s$ and $k_\phi$ are described in the following subsections for different situations.

Stiffness from Buoyancy

An important application of this theory is floating structures. To determine $k_s$ for a beam that floats in water, suppose the beam cross-section has width equal to $b$ at the location where the cross-section penetrates the water surface. Furthermore, let the mass density of water be denoted $\rho_w$, which typically equals $1,025\text{kg/m}^3$ for seawater, and let $g$ denote the acceleration of gravity. For an infinitesimal beam length, $dx$, the change in buoyancy
force due to a vertical displacement $\Delta$ equals the weight of the displaced water, namely weight density multiplied by volume:

$$B = \left( \rho_w g \right) \left( b \cdot dx \cdot \Delta \right)$$

(1)

As a result, the stiffness that resists the displacement $\Delta$, per unit length of the beam, is obtained by dividing $B$ by $dx$ and $\Delta$:

$$k_s = \rho_w g b \frac{kN}{m^2}$$

(2)

which has unit force per unit length squared. The evaluation of $k_s$ according to Eq. (2) is correct as long as the beam width at the waterline does not change. Unfortunately, this condition is not always satisfied, which is understood from Figure 2. While the rectangular beam-cross-section maintains width $b$ regardless of the vertical displacement, the value of $b$ for the circular cross-section varies with the heave motion. As a result, $k_s$ is a function of $\Delta$ for the circular cross-section, which introduces nonlinearity that is neglected in the subsequent equations.

![Figure 2: Heave and roll motion of floating beams.](image)

The fact that the waterline area is only a proxy for displaced water volume becomes even more apparent when determining $k_\phi$. To understand this, first consider the classical approach for determining $k_\phi$. When the rectangular beam-cross-section in Figure 2 rotates counterclockwise as indicated by dashed lines, more water is displaced on the left side and less water is displaced on the right side. As a result, buoyancy forces act against the rotation, giving rise to $k_\phi$. By computing the volumes of the shaded triangles in Figure 2 the resultant buoyancy force, on either side, is

$$R = \left( \rho_w g \right) \frac{1}{2} \frac{b}{2} \cdot \frac{b}{2} \cdot \phi \cdot dx$$

(3)

The distance between the force pair, i.e., the buoyancy forces from each of the shaded triangles in Figure 2 is $2/3$ of $b$. Thus, the moment from buoyancy that resists the rotation is

$$T = R \cdot \frac{2}{3} \cdot b = \rho_w g \cdot \frac{b^3}{12} \cdot \phi \cdot dx$$

(4)
As a result, the stiffness that resists the rotation $\phi$, per unit length of the beam, is obtained by dividing $T$ by $dx$ and $\phi$:

$$k_\phi = \rho_w \cdot g \cdot \frac{b^3}{12} \tag{5}$$

where it is observed that the moment of inertia of the waterline area. Again it is observed that the derivations do not hold for the circular cross-section in Figure 2. In fact, it is easily seen that cross-section has zero resistance against rotation. If not easily seen, it is easily felt when trying to balance on a floating timber log. The problem with applying Eq. (5) to the circular cross-section is the use of movement of the beam at the waterline as a proxy for displaced water volume. Thus, while Eq. (5) is commonly used, it is important to be aware of its limitations.

**Stiffness from Soil**

Suppose the stiffness, $k_s$, is determined from soil testing. In particular, suppose a vertical load, $P$, is placed on an area with dimensions $x$ and $y$, and that the vertical displacement, $\Delta$, is measured. The relationship between the distributed load and the displacement is written in terms of a distributed stiffness, $k_d$:

$$\frac{P}{x \cdot y} = k_d \cdot \Delta \quad \Rightarrow \quad k_d = \frac{P}{\Delta \cdot x \cdot y} \left[ \frac{kN}{m^3} \right] \tag{6}$$

While $k_d$ is stiffness per unit area, $k_s$ is stiffness per unit length. The sought value is obtained by multiplying $k_d$ by the beam width, $b$:

$$k_s = b \cdot k_d \left[ \frac{kN}{m^2} \right] \tag{7}$$

If one assumes that the foundation material is linear elastic, there is no unique relationship between the Young’s modulus, $E$, of the foundation material and the stiffness $k_s$. However, if one imagines that the soil underneath the beam is linear elastic with depth $L$ to bedrock then the force-deformation relationship of the soil is

$$P = \frac{E A}{L} \cdot \Delta \tag{8}$$

where $A = x \cdot y$ is the area loaded by $P$. Writing Eq. (8) in the form of Eq. (6) yields

$$\frac{P}{x \cdot y} = k_d \cdot \Delta \quad \Rightarrow \quad k_d = \frac{E}{L} \left[ \frac{kN}{m^3} \right] \tag{9}$$

and the sought stiffness is, in accordance with Eq. (7):

$$k_s = b \cdot k_d = \frac{b \cdot E}{L} \left[ \frac{kN}{m^2} \right] \tag{10}$$

**Stiffness from Girder Grid**

Another situation appears when the beam is resting on a grid of closely spaced perpendicular beams, e.g. joists. Suppose the joists are spaced at $x$ on centre and that their
stiffness against vertical deflection at the point of intersection with the beam is \( k_b \) [kN/m]. Then the sought stiffness is:

\[
k_s = \frac{k_b}{x} \left[ \frac{kN}{m^2} \right]
\]  

(Differential Equation)

Compared with the basic Euler-Bernoulli beam theory, it is sufficient to modify the equation for vertical equilibrium to obtain the differential equation for a beam on elastic foundation. As a result, the following conventions from basic beam bending hold valid:

1) Clockwise shear force is positive; 2) Bending moment with tension at the bottom is positive; 3) Tension stress is positive; 4) The \( z \)-axis points upwards, so that upwards displacement, \( w \), is positive; 5) The distributed load, \( q_z \), is positive upwards.

Figure 3 shows the forces acting on an infinitesimal beam element. The springs that illustrate the elastic foundation exert a downward force when the beam is subjected to an upward displacement.

Vertical equilibrium yields:

\[
q_z \cdot dx - k_s \cdot w \cdot dx - dV = 0
\]  

(Dividing by \( dx \) and re-arranging yields)

\[
q_z = \frac{dV}{dx} + k_s \cdot w
\]  

(13)

Substitution of this equation into the Euler-Bernoulli beam theory yields the following revised differential equation

\[
EI \cdot \frac{d^4w}{dx^4} + k_s \cdot w = q_z
\]  

(14)
Another way of deriving the differential equation is to start with the following basic differential equation for beam bending:

$$EI \cdot \frac{d^4w}{dx^4} = q_z$$  \hspace{1cm} (15)

From earlier it is understood that the applied load, \(q_z\), plus the elastic foundation yields a total force on the beam element equal to \(q_z - k_s w\). By substituting this total load in the right-hand side of Eq. (15), Eq. (14) is obtained. For convenience, it is rewritten on the form

$$\frac{d^4w}{dx^4} + \frac{k_s}{EI} \cdot w = \frac{q_z}{EI}$$  \hspace{1cm} (16)

In solving this differential equation it is useful to define a “characteristic length,” sometimes referred to as the elastic length. To approach the definition it is first noted that \(\frac{EI}{k_s}\) has dimension \([\text{m}^4]\). As a result, the following definition of the characteristic length has the dimension of length:

$$l_c \equiv \sqrt[4]{\frac{4 \cdot EI}{k_s}}$$  \hspace{1cm} (17)

The convenience of the factor 4 will become apparent later. It is also convenient to work with the normalized coordinate \(\xi\) instead of the original coordinate, \(x\), along the beam:

$$\xi = \frac{x}{l_c}$$  \hspace{1cm} (18)

To transform the differential equation, differentiation with respect to \(x\) is related to differentiation with respect to \(\xi\) by the chain rule of differentiation:

$$\frac{d}{dx} = \frac{d}{d\xi} \cdot \frac{d\xi}{dx} = \frac{1}{l_c} \cdot \frac{d}{d\xi}$$  \hspace{1cm} (19)

This yields the following homogeneous version of the transformed differential equation:

$$\frac{d^4w}{d\xi^4} + 4 \cdot w = 0$$  \hspace{1cm} (20)

**Solution**

The characteristic equation is \(\gamma^4 + 4 = 0\) has the four different complex roots \((1+i), (1-i), (-1+i),\) and \((-1-i)\). Consequently, the general solution is:

$$w(\xi) = C_1 \cdot e^{\frac{\xi}{2}} \cos(\xi) + C_2 \cdot e^{\frac{\xi}{2}} \sin(\xi) + C_3 \cdot e^{\xi} \cos(\xi) + C_4 \cdot e^{\xi} \sin(\xi)$$  \hspace{1cm} (21)

where the phrase “damped terms” is employed to identify terms that vanish as \(\xi\) increases to infinity. This labeling is useful because the solution for a point load must vanish far away from the point of load application. In fact, only the damped terms appear in many
practical situations. To shorten the notation under such circumstances, the following auxiliary functions are defined:

\[ g_1 = e^{-\xi} \cos(\xi) \]
\[ g_2 = e^{-\xi} \sin(\xi) \]
\[ g_3 = g_1 + g_2 \]
\[ g_4 = g_1 - g_2 \]

These functions are plotted in Figure 4, where it is observed that they decay rapidly with \( \xi \). In fact, all functions approach zero once \( \xi \) increases beyond 4.

The auxiliary functions also have the following properties:

\[ \frac{dg_1}{d\xi} = -g_3 \quad \frac{dg_2}{d\xi} = g_4 \quad \frac{dg_3}{d\xi} = -2g_2 \quad \frac{dg_4}{d\xi} = -2g_1 \]

In short, the general solution to the differential equation, with only the damped terms read:

\[ w(\xi) = C_1 \cdot g_1 + C_2 \cdot g_2 \]

The bending moment associated with the solution is:

\[ M = EI \cdot \frac{d^2 w}{dx^2} = \frac{2EI}{l_c^2} \left( C_1 g_2 - C_2 g_1 \right) \]

The shear force is:

\[ V = \frac{dM}{dx} = \frac{2EI}{l_c^3} \left( C_1 g_4 + C_2 g_3 \right) \]
Reference Case 1
Consider the beam in Figure 5, where one end is subjected to the forces $V_o$ and $M_o$, while the other end is “infinitely” far away. Both $V_o$ and $M_o$ are positive, i.e., the shear force is clockwise and the bending moment gives tension at the bottom. Because the beam is infinitely long, the solution cannot have contributions from the un-damped terms. As a result, the solution is given by Eq. (24).

![Figure 5: Reference Case 1.](image)

At $\xi=0$, $g_1=g_3=g_4=1$ and $g_2=0$. As a result, Eqs. (25) and (26) yield the follow expression for the bending moment and shear force at $\xi=0$:

$$M(\xi = 0) = \frac{2EI}{l_c^2} C_2$$  \hspace{1cm} (27)

$$V(x = 0) = \frac{2EI}{l_c^3} (C_1 + C_2)$$  \hspace{1cm} (28)

Setting those expressions equal to $M_o$ and $V_o$, respectively, yields:

$$C_1 = \frac{l_c^2}{2EI} (l_c V_o - M_o)$$  \hspace{1cm} (29)

$$C_2 = \frac{l_c^2}{2EI} M_o$$

Hence, the solution reads:

$$w = -\frac{l_c^2}{2EI} \left( M_o g_3 + l_c V_o g_1 \right)$$

$$\theta = \frac{l_c}{2EI} \left( 2M_o g_1 + l_c V_o g_3 \right)$$  \hspace{1cm} (30)

$$M = M_o g_3 + l_c V_o g_2$$

$$V = -\frac{2}{l_c} M_o g_2 + V_o g_4$$

Reference Case 2
Consider the infinitely long beam in Figure 6, with a point load applied at $\xi=0$. Immediately to the left of the point load the shear force is $P/2$, while immediately to the right it is $-P/2$. That is, with reference to the previous case, $V_o=-P/2$. Furthermore, the rotation at the point load is zero:
\[ \theta(0) = \frac{l_c}{2EI} \left(2M_o + l_c V_o \right) = 0 \]  \hspace{1cm} (31)

Substituting \( V_o = -P/2 \) and solving for \( M_o \) yields

\[ M_o = \frac{l_cP}{4} \]  \hspace{1cm} (32)

where it is noted that the bending moment at the point load is the same as that of a simply supported beam with length \( l_c \) loaded at midspan. In summary, the solution from Case 1 is applicable also here, with

\[ V_o = -\frac{P}{2} \]  \hspace{1cm} (33)

and

\[ M_o = \frac{l_cP}{4} \]  \hspace{1cm} (34)

![Figure 6: Reference Case 2.](image-url)